The second Yamabé invariant with singularities

Mohammed Benalili and Hichem Boughazi

Dedicated to the memory of T. Aubin.

ABSTRACT. Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$.We suppose that g is a metric in the Sobolev space $H_2^p(M, T^*M \otimes T^*M)$ with $p > \frac{n}{2}$ and there exist a point $P \in M$ and $\delta > 0$ such that g is smooth in the ball $B_P(\delta)$. We define the second Yamabe invariant with singularies as the infimum of the second eigenvalue of the singularYamabe operator over a generalized class of conformal metric to g and of volume 1. We show that this operator is attained by a generalized metric, we deduce nodal solutions to a Yamabe type equation with singularities.

1. Introduction

Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. The problem of finding a metric conformal to the original one with constant scalar curvature was first formulated by Yamabe ([**9**]) and a variational method was initiated by this latter in an attempt to solve the problem. Unfortunately or fortunately a serious gap in the Yamabe was pointed out by Trudinger who addressed the question in the case of non positive scalar curvature ([**8**]). Aubin ([**2**]) solved the problem for arbitrary non locally conformally flat manifolds of dimension $n \geq 6$. Finally Shoen ([**7**]) solved completely the problem using the positive-mass theorem founded previously by Shoen himself and Yau . The method to solve the Yamabe problem could be described as follows: let u be a smooth positive function and let $\overline{g} = u^{N-2}g$ be a conformal metric where $N = 2n/(n-2)$. Up to a multiplying constant, the following equation is satisfied

$$L_g(u) = S_{\tilde{g}}|u|^{N-2}u$$

where

$$L_g = \frac{4(n-1)}{n-2}\Delta + S_g$$

1991 *Mathematics Subject Classification.* [].
Key words and phrases. Second Yamabe invariant, singularities, Critical Sobolev growth.

and S_g denotes the scalar curvature of g. L_g is conformally invariant called the conformal operator. Consequently, solving the Yamabe problem is equivalent to find a smooth positive solution to the equation

$$L_g(u) = ku^{N-1} \tag{1}$$

where k is a constant.

In order to obtain solutions to this equation, Yamabe defined the quantity

$$\mu(M,g) = \inf_{u\in C^\infty(M),\ u>0} Y(u)$$

where

$$Y(u)=\frac{\int_M \left(\frac{4(n-1)}{n-2}|\nabla u|^2 + S_g u^2\right) dv_g}{\left(\int_M |u|^N dv_g\right)^{2/N}}.$$

$\mu(M,g)$ is called the Yamabe invariant, and Y the Yamabe functional. In the sequel we write μ instead of $\mu(M,g)$. Writing the Euler-Lagrange equation associated to Y, we see that there exists a one to one correspondence between critical points of Y and solutions of equation (1). In particular, if u is a positive smooth function such that $Y(u)=\mu$, then u is a solution of equation (1) and $\overline{g}=u^{(N-2)}g$ is metric of constant scalar curvature. The key point to solve the Yamabe problem is the following fundamental results due to Aubin ([**2**]). Let S_n be the unit euclidean sphere.

Theorem 1. *Let (M,g) be a compact Riemannian manifold of dimension $n\geq 3$. If $\mu(M,g)<\mu(S_n)$, then there exists a positive smooth solution u such that $Y(u)=\mu(M,g)$.*

This strict inequality $\mu(M,g)<\mu(S_n)$ avoids concentration phenomena. Explicitly $\mu(S_n)=n(n-1)\omega_n^{2/n}$ where ω_n stands for the volume of S_n.

Theorem 2. *Let (M,g) be a compact Riemannian manifold of dimension $n\geq 3$. Then*

$$\mu(M,g)\leq\mu(S_n).$$

Moreover, the equality holds if and only if (M,g) is conformally diffeomorphic to the sphere S_n.

Amman and Humbert ([**1**]) defined the second Yamabe invariant as the infimum of the second eigenvalue of the Yamabe operator over the conformal class of the metric g with volume 1. Their method consists in considering the spectrum of the operator L_g

$$spec(L_g)=\{\lambda_{1,g}\ ,\lambda_{2,g}\ldots\}$$

where the eigenvalues $\lambda_{1,g}<\lambda_{2,g}\ldots$ appear with their multiplicities. The variational characterization of $\lambda_{1,g}$ is given by

$$\lambda_{1,g} = \inf_{u\in C^\infty(M),\ u>0}\frac{\int_M \left(\frac{4(n-1)}{n-2}|\nabla u|^2 + S_g u^2\right) dv_g}{\int_M u^2 dv_g}.$$

Let

$$[g]=\{u^{N-2}g,\ u\in C^\infty(M),\ u>0\}$$

Then they defined the k^{th} Yamabe invariant with $k \in \mathbb{N}^{\star}$, by

$$\mu_k = \inf_{\overline{g} \in [g]} \lambda_{k,\overline{g}} Vol(M, \tilde{g})^{2/n}.$$

With these notations μ_1 is the Yamabe invariant. They studied the second Yamabe invariant μ_2, they found that contrary to the Yamabe invariant, μ_2 cannot be attained by a regular metric. In other words, there does not exist $\overline{g} \in [g]$, such that

$$\mu_2 = \lambda_{2,\overline{g}} Vol(M, \tilde{g})^{2/n}.$$

In order to find minimizers, they enlarged the conformal class to a larger one. A generalized metric is the one of the form $\overline{g} = u^{N-2} g$, which is not necessarily positive and smooth, but only $u \in L^N(M)$, $u \geq 0, u \neq 0$ and where $N = 2n/(n-2)$. The definitions of $\lambda_{2,\overline{g}}$ and of $Vol(M, \overline{g})^{2/n}$ can be extended to generalized metrics. The key points to solve this problem is the following theorems ([**1**]).

Theorem 3. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, then μ_2 is attained by a generalized metrics in the following cases.*

$$\mu > 0 \ , \ \mu_2 < \left[(\mu^{n/2} + (\mu(S_n))^{n/2} \right]^{2/n}$$

and

$$\mu = 0, \quad \mu_2 < \ \mu(S_n)$$

Theorem 4. *The assumptions of the last theorem are satisfied in the following cases*

If (M, g) in not locally conformally flat and, $n \geq 11$ and $\mu > 0$
If (M, g) in not locally conformally flat and, $\mu = 0$ and $n \geq 9$.

Theorem 5. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, assume that μ_2 is attained by a generalized metric $\tilde{g} = u^{N-2} g$ then there exist a nodal solution $w \in C^{2,\alpha}(M)$ of equation*

$$L_g(w) = \mu_2 |u|^{N-2} w$$

such that

$$|w| = u$$

where $\alpha \leq N-2$.

In ([**5**]), recently F.Madani studied the Yamabe problem with singularities when the metric g admits a finite number of points with singularities and smooth outside these points. Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, assume that *g is a metric in the Sobolev space $H_2^p(M, T^*M \otimes T^*M)$ with $p > \frac{n}{2}$ and there exist a point $P \in M$ and $\delta > 0$ such that g is smooth in the ball $B_p(\delta)$, and let (H) be these assumptions.* By Sobolev's embedding, we have for $p > \frac{n}{2}$, $H_2^p(M, T^*M \otimes T^*M) \subset C^{1-[n/p],\beta}(M, T^*M \otimes T^*M)$, where $[n/p]$ denotes the entire part of n/p. Hence the metric satisfying assumption (H) is of class $C^{1-[\frac{n}{p}],\beta}$ with $\beta \in (0,1)$ provided that $p > n$. The Christoffels symbols belong to $H_1^p(M)$ (to $C^o(M)$ in case $p > n$), the Riemannian curvature tensor, the Ricci tensor and scalar curvature are in $L^p(M)$. F. Madani proved under the assumption (H) the existence of a metric $\overline{g} = u^{N-2} g$ conformal to g such that $u \in H_2^p(M)$, $u > 0$ and the scalar curvature $S_{\overline{g}}$ of $\overline{g}$ is constant and (M, g) is not conformal

to the round sphere. Madani proceeded as follows: let $u \in H_2^p(M)$, $u > 0$ be a function and $\overline{g} = u^{N-2}g$ a particular conformal metric where $N = 2n/(n-2)$.Then, multiplying u by a constant, the following equation is satisfied

$$L_g u = \frac{n-2}{4(n-1)} S_{\tilde{g}} |u|^{N-2} u$$

where

$$L_g = \Delta_g + \frac{n-2}{4(n-1)} S_g$$

and the scalar curvature S_g is in $L^p(M)$. Moreover L_g is weakly conformally invariant hence solving the singular Yamabe problem is equivalent to find a positive solution $u \in H_2^p(M)$ of

$$L_g u = k|u|^{N-2} u \tag{2}$$

where k is a constant. In order to obtain solutions of equation (2) we define the quantity

$$\mu = \inf_{u \in H_2^p(M), \quad u>0} Y(u)$$

where

$$Y(u) = \frac{\int_M \left(|\nabla u|^2 + \frac{(n-2)}{4(n-1)} S_g u^2 \right) dv_g}{(\int_M |u|^N dv_g)^{2/N}}.$$

μ is called Yamabe invariant with singularities. Writing the Euler-Lagrange equation associated to Y , we see that there exists a one to one correspondence between critical points of Y and solutions of equation (2). In particular, if $u \in H_2^p$ is a positive function which minimizes Y, then u is a solution of equation(2) and $\overline{g} = u^{N-2}g$ is a metric of constant scalar curvature and μ is attained by a particular conformal metric. The key points to solve the above problem are the following theorems ([**5**]).

Theorem 6. *If $p > n/2$ and $\mu < K^{-2}$then equation2 admits a positive solution $u \in H_2^p(M) \subset C^{1-[n/p],\beta}(M)$; $[n/p]$ is the integer part of n/p, $\beta \in (0,1)$ which minimizes Y, where $K^2 = \frac{4}{n(n-1)} \omega_n^{-2/n}$ with ω_n denotes the volume of S_n. If $p > n$, then $u \in H_2^p(M) \subset C^1(M)$.*

Theorem 7. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. g is a metric which satisfies the assumption (H). If (M, g) is not conformal to the sphere S_n with the standard Riemannian structure then*

$$\mu < K^{-2}$$

Theorem 8. *([**5**]) On an n -dimensional compact Riemannian manifold (M, g), if $u \geq 0$ is a non trivial weak solution in $H_1^2(M)$ of equation $\Delta u + hu = 0$, with $h \in L^p(M)$ and $p > n/2$, then $u \in C^{1-[n/p],\beta}$ and $u > 0$; $[n/p]$ is the integer part of n/p and $\beta \in (0,1)$.*

For regularity argument we need the following results

LEMMA 1. *Let $u \in L_+^N(M)$ and $v \in H_1^2(M)$ a weak solution to $L_g(v) = u^{N-2}v$, then*

$$v \in L^{N+\epsilon}(M)$$

for some $\varepsilon > 0$.

The proof is the same as in ([**5**]) with some modifications. As a consequence of Lemma 7, $v \in L^s(M)$, $\forall s \geq 1$.

PROPOSITION 1. *If $g \in H_2^p(M, T^*M \otimes T^*M)$ is a Riemannian metric on M with $p > n/2$. If $\overline{g} = u^{N-2}g$ is a conformal metric to g such that $u \in H_2^p(M)$, $u > 0$ then L_g is weakly conformally invariant, which means that $\forall v \in H_1^2(M)$, $|u|^{N-1}L_{\overline{g}}(v) = L_g(uv)$ weakly. Moreover if $\mu > 0$, then L_g is coercive and invertible.*

In this paper, let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. We suppose that g is a metric in the Sobolev space $H_2^p(M, T^*M \otimes T^*M)$ with $p > n/2$ and there exist a point $P \in M$ and $\delta > 0$ such that g is smooth in the ball $B_P(\delta)$ and we call these assumptions the condition (H).

In the smooth case the operator L_g is an elliptic operator on M self- adjoint, and has a discrete spectrum $Spec(L_g) = \{\lambda_{1,g}, \lambda_{2,g}, \ldots\}$, where the eigenvalues $\lambda_{1,g} < \lambda_{2,g} \ldots$ appear with their multiplicities. These properties remain valid also in the case where $S_g \in L^p(M)$. The variational characterization of $\lambda_{1,g}$ is given by

$$\lambda_{1,g} = \inf_{u \in H_1^2, u > 0} \frac{\int_M \left(|\nabla u|^2 + \frac{(n-2)}{4(n-1)} S_g u^2 \right) dv_g}{\int_M u^2 dv_g}$$

Let $[g] = \{u^{N-2}g : u \in H_2^p \text{ and } u > 0\}$, Let $k \in \mathbb{N}^*$, we define the k^{th}Yamabe invariant with singularities μ_k as

$$\mu_k = \inf_{\overline{g} \in [g]} \lambda_{k,\overline{g}} Vol(M, \tilde{g})^{2/n}$$

with these notations, μ_1 is the first Yamabe invariant with singularities.

In this work we are concerned with μ_2. In order to find minimizers to μ_2 we extend the conformal class to a larger one consisting of metrics of the form $\overline{g} = u^{N-2}g$ where u is no longer necessarily in $H_2^p(M)$ and positive but $u \in L_+^N(M) = \left\{L^N(M), u \geq 0, u \neq 0\right\}$ such metrics will be called for brevity generalized metrics. First we are going to show that if the singular Yamabe invariant $\mu \geq 0$ then μ_1 it is exactly μ next we consider μ_2, μ_2 is attained by a conformal generalized metric.

Our main results state as follows:

THEOREM 9. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ .We suppose that g is a metric in the Sobolev space $H_2^p(M, T^*M \otimes T^*M)$ with $p > n/2$. There exist a point $P \in M$ and $\delta > 0$ such that g is smooth in the ball $B_P(\delta)$, then*

$$\mu_1 = \mu.$$

Theorem 10. *Let* (M,g) *be a compact Riemannian manifold of dimension* $n\geq 3$*, we suppose that* g *is a metric in the Sobolev space* $H_2^p(M,T^*M\otimes T^*M)$ *with* $p>n/2$ *.There exist a point* $P\in M$ *and* $\delta>0$ *such that* g *is smooth in the ball* $B_P(\delta)$ *.Assume that* μ_2 *is attained by a metric* $\overline{g}=u^{N-2}g$ *where* $u\in L_+^N(M)$*, then there exist a nodal solution* $w\in C^{1-[n/p],\beta}$*,* $\beta\in(0,1)$*, of equation*

$$L_g w=\mu_2 u^{N-2}w\ .$$

Moreover there exist real numbers $a,b>0$ *such that*

$$u=aw_+ + bw_-$$

with $w_+=\sup(w,0)$ *and* $w_-=\sup(-w,0)$ *.*

Theorem 11. *Let* (M,g) *be a compact Riemannian manifold of dimension* $n\geq 3$*, suppose that* g *is a metric in the Sobolev space* $H_2^p(M,T^*M\otimes T^*M)$ *with* $p>n/2$*. There exist a point* $P\in M$ *and* $\delta>0$ *such that* g *is smooth in the ball* $B_P(\delta)$ *then* μ_2 *is attained by a generalized metric in the following cases:*
If (M,g) *in not locally conformally flat and,* $n\geq 11$ *and* $\mu>0$
If (M,g) *in not locally conformally flat and,* $\mu=0$ *and* $n\geq 9$*.*

2. Generalized metrics and the Euler-Lagrange equation

Let

$$L_+^N(M)=\left\{u\in L_+^N(M)\colon\ u\geq 0, u\neq 0\right\}$$

where $N=\frac{2n}{n-2}$.
As in ([**1**])

Definition 1. *For all* $u\in L_+^N(M)$*, we define* $Gr_k^u(H_1^2(M))$ *to be the set of all* $k-$*dimensional subspaces of* $H_1^2(M)$ *with span*$(v_1,v_2,...,v_k)\in Gr_k^u(H_1^2(M))$ *if and only if* $v_1,v_2,...,v_k$ *are linearly independent on* $M-u^{-1}(0)$*.*

Let (M,g) be a compact Riemannian manifold of dimension $n\geq 3$. For a generalized metric $\overline{g}$ conformal to g, we define

$$\lambda_{k,\overline{g}}=\inf_{V\in Gr_k^u(H_1^2(M))}\sup_{v\in V}\frac{\int_M vL_g(v)dv_g}{\int_M |u|^{N-2}v^2dv_g}.$$

We quote the following regularity theorem

Theorem 12. [**7**] *On a* n *-dimensional compact Riemannian manifold* (M,g)*, if* $u\geq 0$ *is a non trivial weak solution in* $H_1^2(M)$ *of the equation*

$$\Delta u+hu=cu^{N-1}$$

with $h\in L^p(M)$ *and* $p>n/2$*, then*

$$u\in H_2^p(M)\subset C^{1-[n/p],\beta}(M)$$

and $u>0$*, where* $[n/p]$ *denotes the integer part of* n/p *and* $\beta\in(0,1)$*.*

Proposition 2. *Let* (v_m) *be a sequence in* $H_1^2(M)$ *such that* $v_m\to v$ *strongly in* $L^2(M)$*, then for all any* $u\in L_+^N(M)$

$$\int_M u^{N-2}(v^2-v_m^2)\ dv_g\to 0.$$

Proof. The proof is the same as in ([**3**]). $\square$

Proposition 3. *If* $\mu > 0$, *then for all* $u \in L_+^N(M)$, *there exist two functions* v, w *in* $H_1^2(M)$ *with* $v \geq 0$ *satisfying in the weak sense the equations*

$$L_g v = \lambda_{1,\overline{g}} u^{N-2} v \tag{7}$$

and

$$L_g w = \lambda_{2,\overline{g}} u^{N-2} w \tag{8}$$

Moreover we can choose v *and* w *such that*

$$\int_M u^{N-2} w^2 dv_g = \int_M u^{N-2} v^2 dv_g = 1 \quad \text{and} \quad \int_M u^{N-2} wv dv_g = 0. \tag{9}$$

Proof. Let $(v_m)_m$ be a minimizing sequence for $\lambda_{1,\tilde{g}}$ i.e. a sequence $v_m \in H_1^2$ such that

$$\lim_m \frac{\int_M v_m L_g(v_m) dv_g}{\int_M |u|^{N-2} v_m^2 dv_g} = \lambda_{1,\tilde{g}}$$

It is well know that $(|v_m|)_m$ is also minimizing sequence. Hence we can assume that $v_m \geq 0$. If we normalize $(v_m)_m$ by

$$\int_M |u|^{N-2} v_m^2 dv_g = 1.$$

Now by the fact that L_g is coercive

$$c \|v_m\|_{H_1^2} \leq \int_M v_m L_g(v_m) dv_g \leq \lambda_{1,\tilde{g}} + 1.$$

$(v_m)_m$ is bounded in $H_1^2(M)$ and after restriction to a subsequence we may assume that there exist $v \in H_1^2(M)$, $v \geq 0$ such that $v_m \to v$ weakly in $H_1^2(M)$, strongly in $L^2(M)$ and almost everywhere in M, then v satisfies in the sense of distributions

$$L_g v = \lambda_{1,\overline{g}} u^{N-2} v.$$

If $u \in H_2^p(M) \subset C^{1-\left[\frac{n}{p}\right],\beta}$ then

$$\int_M u^{N-2} (v^2 - v_m^2) \, dv_g \to 0$$

and

$$\int_M u^{N-2} v^2 dv_g = 1.$$

Then v is not trivial and nonnegative minimizer of $\lambda_{1,\overline{g}}$, by Lemma 7

$$h = S_g - \lambda_{1,\overline{g}} u^{N-2} \in L^p(M)$$

and by Theorem 8

$$v \in C^{1-\left[\frac{n}{p}\right],\beta}(M)$$

and

$$v > 0.$$

If $u \in L_+^N(M)$, by Proposition 2 , we get

$$\int_M u^{N-2} (v^2 - v_m^2) \, dv_g \to 0$$

so

$$\int_M u^{N-2} v^2 dv_g = 1.$$

v is a non negative minimizer in H_1^2 of $\lambda_{1,\overline{g}}$ such that $\int_M u^{N-2} v^2 dv_g = 1$.

Now consider the set

$$E = \{w \in H_1^2\text{: such that } u^{\frac{N-2}{2}} w \neq 0 \text{ and } \int_M u^{N-2} wv dv_g = 0\}$$

and define

$$\lambda'_{2,g} = \inf_{w \in E} \frac{\int_M wL_g(w) dv_g}{\int_M |u|^{N-2} w^2 dv_g}.$$

Let (w_m) be a minimizing sequence for $\lambda'_{2,g}$ i.e. a sequence $w_m \in E$ such that

$$\lim_m \frac{\int_M w_m L_g(w_m) dv_g}{\int_M |u|^{N-2} w_m^2 dv_g} = \lambda'_{2,g}.$$

The same arguments lead to a minimizer w to $\lambda'_{2,g}$with $\int_M u^{N-2} w^2 = 1$.

Now writing

$$\int_M u^{N-2} wv dv_g = \int_M u^{N-2} v(w - w_m) dv_g + \int_M u^{N-2} w_m v dv_g$$

and taking account of $\int_M u^{N-2} w_m v dv_g = 0$ and the fact that $w_m \to w$ weakly in $L^N(M)$ and since $u^{N-2} v \in L^{\frac{N}{N-1}}(M)$, we infer that

$$\int_M u^{N-2} wv dv_g = 0.$$

Hence (8) and (9) are that satisfied with $\lambda'_{2,g}$ instead of $\lambda_{2,\overline{g}}$. □

Proposition 4. *We have*

$$\lambda'_{2,g} = \lambda_{2,\overline{g}}.$$

.

Proof. The Proof is the same as in ([**3**]) so we omit it. □

Remark 1. *If $p > n$ then $u \in H_2^p(M) \subset C^1(M)$, by Theorem 9, v and $w \in C^1(M)$ with $v > 0$.*

Remark 2. *If $p > n$ then $u \in H_2^p(M) \subset C^1(M)$ and $\lambda_{2,\overline{g}} = \lambda_{1,\overline{g}}$, we see that $|w|$ is a minimizer for the functional associated to $\lambda_{1,\overline{g}}$, then $|w|$ satisfies the same equation as v and by Theorem 9 we get $|w| > 0$, this contradicts relation (9) necessarily*

$$\lambda_{2,\overline{g}} > \lambda_{1,\overline{g}}.$$

3. Variational characterization and existence of μ_1

In this section we need the following results

THEOREM 13. *Let (M,g) be a compact n -dimensional Riemannian manifold. For any $\varepsilon>0$, there exists $A(\varepsilon)>0$ such that $\forall u\in H_1^2(M)$,*

$$\|u\|_N^2\le (K^2+\varepsilon)\|\nabla u\|_2^2+A(\varepsilon)\|u\|_2^2$$

where $N=2n/(n-4)$ and $K^2=4/(n(n-2))\ \omega_n^{\frac{-2}{n}}$. ω_n is the volume of the round sphere S_n.

Let $[g]=\{u^{N-2}g:\ u\in H_2^p(M)$ and $u>0\}$, we define the first singular Yamabe invariant μ_1 as

$$\mu_1=\inf_{\overline{g}\in[g]}\lambda_{1,\overline{g}}Vol(M,\tilde{g})^{2/n}$$

then we get

$$\mu_1=\inf_{u\in H_2^p,V\in Gr_1^u(H_1^2)}\sup_{v\in V}\frac{\int_M vL_g(v)dv_g}{\int_M |u|^{N-2}v^2dv_g}(\int_M u^Ndv_g)^{\frac{2}{n}}.$$

LEMMA 2. *We have*

$$\mu_1\le\mu<K^{-2}.$$

.

PROOF. If $p\ge 2n/(n+2)$, the embedding $H_2^p(M)\subset H_1^2\ (M)$ is true, so

$$\mu_1=\inf_{u\in H_2^p,V\in Gr_1^u(H_1^2(M))}\sup_{v\in V}\frac{\int_M vL_g(v)dv_g}{\int_M |u|^{N-2}v^2dv_g}(\int_M u^Ndv_g)^{\frac{2}{n}}$$

$$\le\inf_{u\in H_2^p,V\in Gr_1^u(H_2^p(M))}\sup_{v\in V}\frac{\int_M vL_g(v)dv_g}{\int_M |u|^{N-2}v^2dv_g}(\int_M u^Ndv_g)^{\frac{2}{n}}.$$

in particular for $p>\frac{n}{2}$ and $u=v$ we get

$$\mu_1\le\inf_{v\in H_2^P,V\in Gr_1^u(H_2^P(M))}\sup_{v\in V}\frac{\int_M vL_g(v)dv_g}{\int_M |v|^{N-2}v^2dv_g}(\int_M v^Ndv_g)^{\frac{2}{n}}=\mu$$

i.e

$$\mu_1\le\mu<K^{-2}.$$

□

THEOREM 14. *If $\mu>0$, there exits conform metric $\overline{g}=u^{N-2}g$ which minimizes μ_1.*

PROOF. The proof will take several steps.
Step1.
We study a sequence of metrics $g_m=u_m^{N-2}g$ with $u_m\in H_2^p(M)$, $u_m>0$ which minimize μ_1 i.e. a sequence of metrics such that

$$\mu_1=\lim_m\lambda_{1,m}(Vol(M,g_m))^{2/n}.$$

Without loss of generality, we may assume that $Vol(M, g_m) = 1$ i.e.

$$\int_M u_m^N dv_g = 1.$$

In particular, the sequence of functions u_m is bounded in $L^N(M)$ and there exists $u \in L^N(M)$, $u \geq 0$ such that $u_m \to u$ weakly in $L^N(M)$. We are going to prove that the generalized metric $u^{N-2}g$ minimizes μ_1. Proposition 3 implies the existence of a sequence (v_m) of class $H_1^2(M)$, $v_m > 0$ such that

$$L_g(v_m) = \lambda_{1,m} u_m^{N-2} v_m$$

and

$$\int_M u_m^{N-2} v_m^2 dv_g = 1.$$

now since $\mu > 0$, by Proposition 1, L_g is coercive and we infer that

$$c\|v_m\|_{H_1^2} \leq \int_M v_m L_g(v_m) dv_g = \lambda_{1,m} \leq \mu_1 + 1.$$

The sequence $(v_m)_m$ is bounded in $H_1^2(M)$, we can find $v \in H_1^2(M)$, $v \geq 0$ such that $v_m \to v$ weakly in $H_1^2(M)$. Together with the weak convergence of $(u_m)_m$, we obtain in the sense of distributions

$$L_g(v) = \mu_1 u^{N-2} v.$$

Step2.

Now we are going to show that $v_m \to v$ strongly in $H_1^2(M)$.

We put

$$z_m = v_m - v$$

then $z_m \to 0$ weakly in $H_1^2(M)$ and strongly in $L^q(M)$ with $q < N$, and writing

$$\int_M |\nabla v_m|^2 dv_g = \int_M |\nabla z_m|^2 dv_g + \int_M |\nabla v|^2 dv_g + 2\int_M \nabla z_m \nabla v dv_g$$

we see that

$$\int_M |\nabla v_m|^2 dv_g = \int_M |\nabla z_m|^2 dv_g + \int_M |\nabla v|^2 dv_g + o(1).$$

Now because of $2p/(p-1) < N$, we have

$$\int_M \frac{n-2}{4(n-1)} S_g (v_m - v)^2 dv_g \leq \frac{n-2}{4(n-1)} \|S_g\|_p \|v_m - v\|^2_{\frac{2p}{p-1}} \to 0$$

so

$$\int_M \frac{n-2}{4(n-1)} S_g {v_m}^2 dv_g = \int_M \frac{n-2}{4(n-1)} S_g v^2 dv_g + 0(1)$$

and

$$\int_M |\nabla v_m|^2 dv_g + \int_M \frac{n-2}{4(n-1)} S_g (v_m)^2 dv_g$$

$$= \int_M |\nabla z_m|^2 dv_g + \int_M |\nabla v|^2 dv_g + \int_M \frac{n-2}{4(n-1)} S_g (v)^2 dv_g + 0(1).$$

Then

$$\int_M v_m L_g v_m dv_g = \int_M |\nabla z_m|^2 dv_g + \int_M |\nabla v|^2 dv_g + \int_M \frac{n-2}{4(n-1)} S_g v^2 dv_g + 0(1)$$

And by the definition of μ and Lemma 2 we get

$$\int_M |\nabla v|^2 dv_g + \int_M \frac{n-2}{4(n-1)} S_g(v)^2 dv_g \geq \mu(\int_M v^N dv_g)^{\frac{2}{N}} \geq \mu_1(\int_M v^N dv_g)^{\frac{2}{N}}$$

then

$$\int_M v_m L_g(v_m) dv_g \geq \int_M |\nabla z_m|^2 dv_g + \mu_1(\int_M v^N dv_g)^{\frac{2}{N}} + 0(1).$$

And since

$$\int_M v_m L_g(v_m) dv_g = \lambda_{1,m} \leq \mu_1 + 0(1)$$

and

$$\int_M |\nabla z_m|^2 dv_g + \mu_1(\int_M v^N dv_g)^{\frac{2}{N}} \leq \mu_1 + 0(1)$$

i.e

$$\mu_1 \|v\|_N^2 + \|\nabla z_m\|_2^2 \leq \mu_1 + 0(1) \tag{10}$$

Now by Brezis-Lieb lemma, we get

$$\lim_m \int_M v_m^N + z_m^N dv_g = \int_M v^N dv_g$$

i.e.

$$\lim_m \|v_m\|_N^N - \|z_m\|_N^N = \|v\|_N^N.$$

Hence

$$\|v_m\|_N^N + 0(1) = \|z_m\|_N^N + \|v\|_N^N.$$

By Hölder's inequality and $\int_M u_m^{N-2} v_m^2 dv_g = 1$, we get

$$\|v_m\|_N^N \geq 1$$

i.e.

$$\int_M v^N + z_m^N \, dv_g = \int_M v_m^N dv_g + 0(1) \geq 1 + 0(1).$$

Then

$$\left(\int_M v^N dv_g\right)^{\frac{2}{N}} + \left(\int_M z_m^N dv_g\right)^{\frac{2}{N}} \geq 1 + 0(1)$$

i.e.

$$\|z_m\|_N^2 + \|v\|_N^2 \geq 1 + 0(1).$$

Now by Theorem 13 and the fact $z_m \to 0$ strongly in L^2, we get

$$\|z_m\|_N^2 \leq (K^2 + \varepsilon)\|\nabla z_m\|_2^2 + o(1)$$

$$1 + o(1) \leq \|z_m\|_N^2 + \|v\|_N^2 \leq \|v\|_N^2 + (K^2 + \varepsilon)\|\nabla z_m\|_2^2 + o(1).$$

So we deduce

$$1+o(1)\leq \|v\|_N^2+(K^2+\varepsilon)\|\nabla z_m\|_2^2+o(1)$$

and from inequality (10), we get

$$\|\nabla z_m\|_2^2+\mu_1\|v\|_N^2\leq \mu_1((K^2+\varepsilon)\|\nabla z_m\|_2^2+\|v\|_N^2)+o(1).$$

So if $\mu_1 K^2<1$, we get

$$(1-\mu_1(K^2+\varepsilon))\|\nabla z_m\|_2^2)\leq 0(1)$$

i.e. $v_m\to v$ strongly in $H_1^2(M)$.

Step3. We have

$$\lim_m \int_M \left(u_m^{N-2}v_m^2-u^{N-2}v^2+u_m^{N-2}v^2-u_m^{N-2}v^2\right)dv_g$$
$$=\lim_m \int \left(u_m^{N-2}(v_m^2-v^2)+(u_m^{N-2}-u^{N-2})v^2\right)dv_g.$$

Now since $u_m\to u$ a.e. so does $u_m^{N-2}\to u^{\ N-2}$ and $\int_M u_m^{N-2}dv_g\leq c$, hence u_m^{N-2} is bounded in $L^{N/(N-2)}$ and up to a subsequence $u_m^{N-2}\to u^{\ N-2}$ weakly in $L^{N/(N-2)}$. Because of $v^2\in L^{\frac{N}{2}}(M)$, we have

$$\lim_m \int (u_m^{N-2}-u^{N-2})v^2dv_g=0$$

and by Hölder's inequality

$$\lim_m \int u_m^{N-2}(v_m-v)^2dv_g\leq (\int u_m^N dv_g)^{(N-2)/N}(\int_m |v_m-v|^N dv_g)^{\frac{2}{N}}\leq 0.$$

By the strong convergence of v_m in $L^N(M)$, we get $\int_M u^{N-2}v^2dv_g=1$, then v and u are non trivial functions.

Step4.

Let $\overline{u}=av\in L_+^N(M)$ with $a>0$ a constant such that $\int\limits_M \overline{u}^N dv_g=1$ with v a solution of

$$L_g(v)=\mu_1 u^{N-2}v$$

with the constraint

$$\int_M u^{N-2}v^2dv_g=1.$$

We claim that $u=v$; indeed,

$$\mu_1\leq \frac{\int_M vL_g(v)dv_g}{\int_M \overline{u}^{N-2}v^2dv_g}$$
$$\leq \frac{\int_M vL_g(v)dv_g}{\int_M (av)^{N-2}v^2dv_g}=\frac{a^2\mu_1\int_M u^{N-2}v^2dv_g}{\int_M \overline{u}^{N-2}(av)^2dv_g}$$

and Hölder's inequality lead

$$\leq \mu_1\int_M (u)^{N-2}(av)^2dv_g$$

$$\leq \mu_1(\int_M (u)^{N-2\frac{N}{N-2}})^{\frac{N-2}{N}}(\int_M (av)^{2\frac{N}{2}} dv_g)^{\frac{2}{N}} \leq \mu_1.$$

And since the equality in Hölder's inequality holds if

$$\overline{u} = u = av$$

then $a = 1$ and

$$u = v \ .$$

Then v satisfies $L_g v = \mu_1 v^{N-1}$, by Theorem 12 we get $v = u \in H_2^p(M) \subset C^{1-[\frac{n}{p}],\beta}(M)$ with $\beta \in (0,1)$ and $v = u > 0$,

Resuming, we have

$$L_g(v) = \mu_1 v^{N-1} \ , \quad \int_M v^N dv_g = 1 \ \text{ and } \ v = u \in H_2^p(M) \subset C^{1-[\frac{n}{p}],\beta}(M)$$

so the metric $\tilde{g} = u^{N-2} g$ minimizes μ_1. □

4. Yamabe conformal invariant with singularities

Theorem 15. *If* $\mu \geq 0$*, then* $\mu_1 = \mu$

Proof. Step1

If $\mu > 0$. Let v such that $L_g(v) = \mu_1 v^{N-1}$ and $\int_M v^N dv_g = 1$ then

$$\mu_1 = \int_M v L_g(v) dv_g \geq c \|v\|_{H_1^2}$$

and v in non trivial function then $\mu_1 > 0$. On the other hand

$$\mu = \inf \frac{\int_M v L_g(v) dv_g}{(\int_M v^N dv_g)^{\frac{2}{N}}}$$
$$\leq \int_M v L_g(v) dv_g = \mu_1$$

and by Lemma 2 , we get

$$\mu_1 = \mu$$

Step2

If $\mu = 0$, Lemma 2 implies that $\mu_1 \leq 0$, hence

$$\mu_1 = 0.$$

□

5. Variational characterization of μ_2

Let $[g] = \{u^{N-2} g, \ \ u \in H_2^p(M) \text{ and } u > 0\}$, we define the second Yamabe invariant μ_2 as

$$\mu_2 = \inf_{\overline{g} \in [g]} \lambda_{2,\overline{g}} Vol(M, \overline{g})^{2/n}$$

or more explicitly

$$\mu_2 = \inf_{u \in H_2^P, V \in Gr_2^u(H_1^2(M))} \sup_{v \in V} \frac{\int_M v L_g(v) dv_g}{\int_M |u|^{N-2} v^2 dv_g} (\int_M u^N dv_g)^{\frac{2}{n}}$$

THEOREM 16. [**1**] *On compact Riemannian manifold* (M,g) *of dimension* $n \geq 3$*, we have for all* $v \in H_1^2(M)$ *and for all* $u \in L_+^N(M)$

$$2^{\frac{2}{n}}\int_M |u|^{N-2}v^2dv_g \leq (K^2\int_M |\nabla v|^2dv_g + \int_M B_0v^2dv_g)(\int_M u^Ndv_g)^{\frac{2}{n}}$$

Or

$$2^{\frac{2}{n}}\int_M |u|^{N-2}v^2dv_g \leq \mu_1(S_n)(\int_M C_n|\nabla v|^2 + B_0v^2dv_g)(\int_M u^Ndv_g)^{\frac{2}{n}}$$

THEOREM 17. [**1**] *For any compact Riemannian manifold* (M,g) *of dimension* $n \geq 3$*, there exists* $B_0 > 0$ *such that*

$$\mu_1(S_n) = n(n-1)\omega_n^{2/n} \ = \inf_{H_1^2}\frac{\int_M \frac{4(n-1)}{(n-2)}|\nabla u|^2 + B_0u^2dv_g}{(\int_M |u|^Ndv_g)^{2/N}}$$

where ω_n *is the volume of the unit round sphere*

or

$$(\int_M |u|^Ndv_g)^{2/N} \leq K^2\int_M |\nabla u|^2dv_g + \int_M B_0u^2dv_g$$

$K^2 = \mu_1(S_n)^{-1}C_n$ *and* $C_n = (4(n-1))/(n-2)$

6. Properties of μ_2

We know that g is smooth in the ball $B_p(\delta)$ by assumption (H), this assumption is sufficient to prove that Aubin's conjecture is valid. The case (M,g) is not conformally flat in a neighborhood of the point P and $n \geq 6$, let η is a cut-off function with support in the ball $B_p(2\varepsilon)$ and $\eta = 1$ in $B_p(\varepsilon)$, where $2\varepsilon \leq \delta$ and

$$v_\varepsilon(q) = (\frac{\varepsilon}{r^2+\varepsilon^2})^{\frac{n-2}{2}}$$

with $r = d(p,q)$.We let $u_\varepsilon = \eta v_\varepsilon$ and define

$$Y(u) = \frac{\int_M \left(|\nabla u|^2 + \frac{n-2}{4(n-1)}S_gu^2\right)dv_g}{(\int_M |u|^Ndv_g)^{2/N}}.$$

We obtain the following lemma

LEMMA 3. [**1**]

$$\mu = Y(v_\varepsilon) \leq \begin{cases} \{(K^{-2} - c|w(P)|^2\varepsilon^4 + 0(\varepsilon^4) & \text{if } n > 6 \\ K^{-2} - c|w(P)|^2\varepsilon^4\log\frac{1}{\varepsilon} + 0(\varepsilon^4) & \text{if } n = 6 \end{cases}$$

where $|w(P)|$*is the norm of the Weyl tensor at the point* P *and* $c > 0$.

THEOREM 18. *If* (M,g) *in not locally conformally flat and* $n \geq 11$ *and* $\mu > 0$ *, we find*

$$\mu_2 < ((\mu^{\frac{n}{2}} + (K^{-2})^{\frac{n}{2}})^{\frac{2}{n}}$$

and if $\mu = 0$ *,* $n \geq 9$ *then*

$$\mu_2 < K^{-2}$$

PROOF. With the same method as in [**1**], this lemma follows from theorem18. ∎

7. Existence of a minimizer to μ_2

LEMMA 4. *Assume that $v_m \to v$ weakly in $H_1^2(M)$, $u_m \to u$ weakly in $L^N(M)$ and $\int_M {u_m}^{N-2}{v_m}^2 dv_g = 1$ then*

$$\int_M {u_m}^{N-2}(v_m - v)^2 dv_g = 1 - \int_M u^{N-2}v^2 dv_g + o(1)$$

PROOF. we have

$$\int_M {u_m}^{N-2}(v_m - v)^2 dv_g$$

$$= \int_M {u_m}^{N-2}{v_m}^2 dv_g + \int_M {u_m}^{N-2}v^2 dv_g - \int_M 2{u_m}^{N-2}v_m v dv_g$$

$$= 1 + \int_M {u_m}^{N-2}v^2 dv_g - \int_M 2{u_m}^{N-2}v_m v dv_g \ . \tag{15}$$

Now $\left({u_m}^{N-2}\right)_m$ is bounded in $L^{\frac{N}{N-2}}(M)$ and ${u_m}^{N-2} \to u^{N-2}$a.e., then ${u_m}^{N-2} \to u^{N-2}$ weakly in $L^{\frac{N}{N-2}}(M)$ and $\forall \phi \in L^{\frac{N}{2}}$

$$\int_M \phi {u_m}^{N-2} dv_g \to \int_M \phi u^{N-2} dv_g$$

in particular for $\phi = v^2$

$$\int_M v^2 {u_m}^{N-2} dv_g \to \int_M v^2 u^{N-2} dv_g.$$

$\int_M {u_m}^{N-2} v_m dv_g$ is bounded in $L^{\frac{N}{N-1}}(M)$, because of

$$\int_M {u_m}^{N-2\frac{N}{N-1}} v_m^{\frac{N}{N-1}} dv_g \leq (\int_M {u_m}^N dv_g)^{\frac{N-2}{N-1}} (\int_M v_m^N dv_g)^{\frac{1}{N-1}}$$

and ${u_m}^{N-2}v_m \to u^{N-2}v$ a.e., then ${u_m}^{N-2}v_m \to u^{N-2}v$ weakly in $L^{\frac{N}{N-1}}(M)$. Hence

$$\int_M {u_m}^{N-2} v_m v dv_g \to \int_M u^{N-2}v^2 dv_g$$

and

$$\int_M {u_m}^{N-2}(v_m - v)^2 dv_g = 1 - \int_M u^{N-2}v^2 dv_g + o(1).$$

□

THEOREM 19. *If $1 - 2^{-\frac{2}{n}}K^2\mu_2 > 0$, then the generalized metric $u^{N-2}g$ minimizes μ_2*

PROOF. *Step1.*

We study a sequence of metrics $g_m = u_m^{N-2}g$ with $u_m \in H_2^p(M)$, $u_m > 0$ which minimizes the infimum in the definition of μ_2 i.e. a sequence of metrics such that

$$\mu_2 = \lim \lambda_{2,m}(Vol(M, g_m))^{2/n}.$$

Without loss generality, we may assume that $Vol(M, g_m) = 1$ i.e. that $\int_M u_m^N dv_g =$ 1.In particular, the sequence of functions $(u_m)_m$ is bounded in $L^N(M)$ and there

exists $u\in L^{N}(M)$, $u\geq 0$ such that $u_{m}\rightarrow u$ weakly in L^{N}. We are going to prove that the generalized metric $u^{N-2}g$ minimizes μ_{2} .Proposition 3 , implies the existence of $v_{m},w_{m}\in H_{1}^{2}(M)$, $v_{m}>0$ such that

$$L_g(v_m)=\lambda_{1,m}u_m^{N-2}v_m$$

$$L_g(w_m)=\lambda_{2,m}u_m^{N-2}w_m$$

And such that

$$\int_M u_m^{N-2}v_m^2dv_g=\int_M u_m^{N-2}w_m^2dv_g=1,\int_M u_m^{N-2}v_mw_mdv_g=0.$$

The sequence v_{m} ,w_{m} is bounded in $H_{1}^{2}(M)$, we can find $v,w\in H_{1}^{2}(M)$, $v\geq 0$ such that $v_{m}\rightarrow v$, $w_{m}\rightarrow w$ weakly in $H_{1}^{2}(M)$.Together with the weak convergence of (u_{m}), we get in weak sense

$$L_g(v)=\widehat{\mu_1}u^{N-2}v$$

and

$$L_g(w)=\mu_2u^{N-2}w$$

where

$$\widehat{\mu_1}=\lim\lambda_{1,m}\leq\ \mu_2.$$

Step2.

Now we show $v_{m}\rightarrow v$, $w_{m}\rightarrow w$ strongly in $H_{1}^{2\cdot}(M)$. Applying Theorem 16 to the sequence $v_{m}-v$, we get

$$\int_M |u_m|^{N-2}(v_m-v)^2dv_g\leq(2^{-\frac{2}{n}}K^2\int_M|\nabla(v_m-v)|^2dv_g+\int_M B_0(v_m-v)^2dv_g)(\int_M u^Ndv_g)^{\frac{2}{n}}$$

and since $v_{m}\rightarrow v$ strongly in L^{2} ,

$$\begin{aligned}\int_M |u_m|^{N-2}(v_m-v)^2dv_g&\leq(2^{-\frac{2}{n}}K^2\int_M|\nabla(v_m-v)|^2dv_g+o(1)\\&\leq(2^{-\frac{2}{n}}K^2\int_M|\nabla(v_m)|^2+|\nabla v|^2-2\nabla v_m\nabla vdv_g+o(1).\end{aligned}$$

By the weak convergence of (v_{m}) , $\int_{M}\nabla v_{m}\nabla vdv_{g}=\int_{M}\left|\nabla v\right|^{2}dv_{g}+o(1)$

$$\int_M |u_m|^{N-2}(v_m-v)^2dv_g\leq(2^{-\frac{2}{n}}K^2\int_M|\nabla(v_m)|^2-\left|\nabla v\right|^2dv_g+o(1)$$

and since

$$\int_M\frac{n-2}{4(n-1)}S_g{v_m}^2dv_g=\int_M\frac{n-2}{4(n-1)}S_gv^2dv_g+0(1)$$

we get

$$\int_M |u_m|^{N-2}(v_m-v)^2dv_g\leq 2^{-\frac{2}{n}}K^2(\int_M|\nabla(v_m)|^2-\left|\nabla v\right|^2dv_g$$

$$+\int_M\frac{n-2}{4(n-1)}S_g(v_m^2-v^2)dv_g)+o(1)\leq 2^{-\frac{2}{n}}K^2(\int_M v_mL_g(v_m)-vL_g(v)dv_g)+o(1)$$

$$\leq 2^{-\frac{2}{n}}K^2(\lambda_{1,m}-\widehat{\mu_1}\int_M u^{N-2}v^2dv_g)+o(1)$$

By the fact $\widehat{\mu_1} = \lim \lambda_{1,m} \leq \mu_2$

$$\leq 2^{-\frac{2}{n}} K^2 \mu_2 (1 - \int_M u^{N-2} v^2 dv_g) + o(1)$$

Then

$$\int_M |u_m|^{N-2} (v_m - v)^2 dv_g \leq 2^{-\frac{2}{n}} K^2 \mu_2 (1 - \int_M u^{N-2} v^2) dv_g + o(1)$$

Now using the weak convergence of (v_m) in $H_1^2(M)$ and the weak convergence of (u_m) in $L^N(M)$, we infer by Lemma 4 that

$$\int_M |u_m|^{N-2} (v_m - v)^2 dv_g = 1 - \int_M u^{N-2} v^2 dv_g + o(1)$$

then

$$1 - \int_M u^{N-2} v^2 dv_g \leq 2^{-\frac{2}{n}} K^2 \mu_2 (1 - \int_M u^{N-2} v^2) dv_g + o(1)$$

and

$$1 - 2^{-\frac{2}{n}} K^2 \mu_2 \leq (1 - 2^{-\frac{2}{n}} K^2 \mu_2) \int_M u^{N-2} v^2 dv_g + o(1).$$

So if $1 - 2^{-\frac{2}{n}} K^2 \mu_2 > 0$ then

$$\int_M u^{N-2} v^2 dv_g \geq 1.$$

and by Fatou's lemma, we obtain

$$\int_M u^{N-2} v^2 dv_g \leq \underline{\lim} \int_M u_m^{N-2} v_m^2 dv_g = 1.$$

We find that

$$\int_M u^{N-2} v^2 dv_g = 1. \tag{16}$$

So u and v are not trivial.

Moreover

$$\int_M |\nabla(v_m - v)|^2 dv_g = \int_M \left(|\nabla(v_m)|^2 + |\nabla v|^2 - 2\nabla v_m \nabla v \right) dv_g$$

$$= \int_M |\nabla(v_m)|^2 - |\nabla v|^2 dv_g + o(1)$$

and since $\int_M S_g \left({v_m}^2 - v^2 \right) dv_g = o(1)$, we get

$$\int_M |\nabla(v_m - v)|^2 dv_g = \int_M v_m L_g(v_m) - v L_g(v) dv_g + o(1)$$

$$\leq \mu_2 (1 - \int_M u^{N-2} v^2 dv_g) + o(1)$$

Then, by relation (16)

$$\int_M |\nabla(v_m - v)|^2 dv_g = o(1)$$

and $v_m \to v$ strongly in $H_1^2(M)$ .The same argument holds with (w_m), hence $w_m \to w$ strongly in $H_1^2(M)$ and $\int_M u^{N-2}w^2 dv_g = 1$.

To show that $\int_M u^{N-2}vwdv_g = 0$, first writing and using Hôlder's inequality, we get

$$\int_M \left(u_m^{N-2}v_m w_m - u^{N-2}vw\right) dv_g = \int_M \left(u_m^{N-2}v_m w_m - u_m^{N-2}vw_m + u_m^{N-2}vw_m - u^{N-2}vw\right) dv_g$$

$$= \int_M u_m^{N-2}(v_m - v)w_m dv_g + \int_M \left(u_m^{N-2}vw_m - u^{N-2}vw\right) dv_g$$

$$= \int_M u_m^{\frac{N-2}{2}} w_m[u_m^{\frac{N-2}{2}}(v_m - v)]dv_g + \int_M \left(u_m^{N-2}vw_m - u^{N-2}vw\right) dv_g$$

$$\leq \left(\int_M u_m^{N-2}w_m^2 dv_g\right)^{\frac{1}{2}} \left(\int_M u_m^{N-2}(v_m - v)^2 dv_g\right)^{\frac{1}{2}} + \int_M \left(u_m^{N-2}vw_m - u^{N-2}vw\right) dv_g$$

$$\leq \left(\int_M u_m^{N-2}(v_m - v)^2 dv_g\right)^{\frac{1}{2}} + \int_M \left(u_m^{N-2}vw_m - u^{N-2}vw\right) dv_g$$

$$\leq \left[\left(\int_M u_m^{N-2\frac{N}{N-2}} dv_g\right)^{\frac{N-2}{N}} \left(\int_M |v_m - v|^N dv_g\right)^{\frac{2}{N}}\right]^{\frac{1}{2}} + \int_M \left(u_m^{N-2}vw_m - u^{N-2}vw\right) dv_g$$

$$\leq \left(\int_M |v_m - v|^N dv_g\right)^{\frac{1}{N}} + \int_M \left(u_m^{N-2}vw_m - u^{N-2}vw\right) dv_g$$

$$\leq \left(\int_M |v_m - v|^N dv_g\right)^{\frac{1}{N}} + \int_M \left(u_m^{N-2}vw_m - u_m^{N-2}vw + u_m^{N-2}vw - u^{N-2}vw\right) dv_g$$

$$\leq \left(\int_M |v_m - v|^N dv_g\right)^{\frac{1}{N}} + \int_M \left(u_m^{N-2}v(w_m - w) + (u_m^{N-2} - u^{N-2})vw\right) dv_g$$

$$\leq \left(\int_M |v_m - v|^N dv_g\right)^{\frac{1}{N}} + \int_M \left((u_m^{\frac{N-2}{2}}v)(u_m^{\frac{N-2}{2}}(w_m - w)) + (u_m^{N-2} - u^{N-2})vw\right) dv_g$$

$$\leq \left(\int_M |v_m - v|^N dv_g\right)^{\frac{1}{N}} + \left(\int_M u_m^{N-2}v^2 dv_g\right)^{\frac{1}{2}} \left(\int_M u_m^{N-2}(w_m - w)^2 dv_g\right)^{\frac{1}{2}} + \int_M (u_m^{N-2} - u^{N-2})vwdv_g$$

$$\leq \left(\int_M |v_m - v|^N dv_g\right)^{\frac{1}{N}} + \left(\int_M u_m^{N-2}v^2 dv_g\right)^{\frac{1}{2}} \left(\int_M |w_m - w|)^N dv_g\right)^{\frac{1}{N}} + \int_M (u_m^{N-2} - u^{N-2})vwdv_g.$$

Now noting that

$$\int_M u_m^{N-2}v^2 dv_g \leq (\int_M u_m^N dv_g)^{\frac{N-2}{2}} (\int_M v^N dv_g)^{\frac{2}{N}} < +\infty$$

and taking account of $u_m^{N-2} \to u^{N-2}$ weakly in $L^{\frac{N}{N-2}}(M)$ and the fact that $vw \in L^{\frac{N}{2}}(M)$, we deduce

$$\int_M (u_m^{N-2} - u^{N-2}) vw dv_g \to 0$$

hence

$$\int_M u^{N-2} vw dv_g = 0.$$

Consequently the generalized metric $u^{N-2}g$ minimizes μ_2. □

THEOREM 20. *If $\mu_2 < K^{-2}$, then generalized metric $u^{N-2}g$ minimizes μ_2*

PROOF. *Step1.*

We study a sequence of metrics $g_m = u_m^{N-2}g$ with $u_m \in H_2^p(M)$, $u_m > 0$ which attained μ_2 i.e. a sequence of metrics such that

$$\mu_2 = \lim_m \lambda_{2,m}(Vol(M, g_m))^{2/n}.$$

Without loss of generality, we may assume that $Vol(M, g_m) = 1$ i.e. $\int_M u_m^N dv_g = 1$.In particular, the sequence $(u_m)_m$ is bounded in $L^N(M)$and there exists $u \in L^N(M)$, $u \geq 0$ such that $u_m \to u$ weakly in $L^N(M)$.We are going to prove that the metric $u^{N-2}g$ minimizes μ_2. Proposition 3 and Theorem 8 imply the existence of $v_m, w_m \in C^{1-\left[\frac{n}{p}\right],\beta}$, with $\beta \in (0,1)(M)$, $v_m > 0$ such that

$$L_g(v_m) = \lambda_{1,m} u_m^{N-2} v_m$$

$$L_g(w_m) = \lambda_{2,m} u_m^{N-2} w_m$$

and

$$\int_M u_m^{N-2} v_m^2 dv_g = \int_M u_m^{N-2} w_m^2 dv_g = 1, \int_M u_m^{N-2} v_m w_m dv_g = 0.$$

The sequences $(v_m)_m$and $(w_m)_m$ are bounded in H_1^2, we can find $v, w \in H_1^2$ with $v \geq 0$ such that $v_m \to v$, $w_m \to w$ weakly in H_1^2.Together with the weak convergence of $(u_m)_m$, we get in the weak sense

$$L_g(v) = \widehat{\mu_1} u^{N-2} v$$

and

$$L_g(w) = \mu_2 u^{N-2} w$$

where

$$\widehat{\mu_1} = \lim \lambda_{1,m} \leq \mu_2.$$

Step2.

Now we are going to show that $v_m \to v$, $w_m \to w$ strongly in $H_1^{2.}$.

By Hölder's inequality, Theorem 13, strong convergence of v_m in $L^{2.}$,we get

$$\int_M |u_m|^{N-2}(v_m - v)^2 dv_g \leq \|v_m - v\|_N^2 \leq K^2 \|\nabla(v_m - v)\|_2^2 + o(1)$$

$$\leq K^2 \int_M |\nabla(v_m)|^2 + |\nabla v|^2 - 2\nabla v_m \nabla v dv_g + o(1)$$

$$\leq K^2 \int_M |\nabla(v_m)|^2 - |\nabla v|^2 \, dv_g + o(1)$$

$$\leq K^2 \int_M v_m L_g(v_m) - v L_g(v) dv_g + o(1)$$

$$\leq K^2 \mu_2 (1 - \int_M u^{N-2} v^2 dv_g) + o(1)$$

and with Lemma 4

$$\int_M |u_m|^{N-2} (v_m - v)^2 dv_g = 1 - \int_M u^{N-2} v^2 dv_g + o(1)$$

then

$$1 - \int_M u^{N-2} v^2 dv_g \leq K^2 \mu_2 (1 - \int_M u^{N-2} v^2 dv_g) + o(1)$$

i.e

$$1 - K^2 \mu_2 \leq (1 - K^2 \mu_2) \int_M u^{N-2} v^2 dv_g$$

so if $1 - K^2 \mu_2 > 0$,

$$\int_M u^{N-2} v^2 dv_g \geq 1.$$

On the other hand since by Fatou's lemma

$$\int_M u^{N-2} v^2 dv_g \leq \underline{\lim} \int_M u_m^{N-2} v_m^2 dv_g = 1.$$

Then

$$\int_M u^{N-2} v^2 dv_g = 1.$$

and

$$\int_M |\nabla(v_m - v)|^2 \, dv_g = o(1)$$

Hence $v_m \to v$ strongly in $H_1^{2.} \subset L^N$.
The same conclusion also holds for $(w_m)_m$. $\square$

LEMMA 5. *Let* $u \in L^N$ *with* $\int_M u^N dv_g = 1$ *and* z , w *nonnegative functions in* H_1^2*satisfying*

$$\int_M w L_g(w) dv_g \leq \mu_2 \int_M u^{N-2} w^2 dv_g \tag{20}$$

and

$$\int_M z L_g(z) dv_g \leq \mu_2 \int_M u^{N-2} z^2 dv_g \tag{21}$$

And suppose that $(M - z^{-1}(0)) \cap (M - w^{-1}(0))$ *has measure zero. Then* u *is a linear combination of* z *and* w *and we have equality in (20) and (21).*

PROOF. The proof is the same as that of Aummann and Humbert in [**1**]. $\square$

THEOREM 21. *If a generalized metric* $u^{N-2}g$ *minimizes* μ_2, *then there exist a nodal solution* $w \in H_2^p \subset C^{1-[n/p],\beta}$
of equation

$$L_g(w) = \mu_2 u^{N-2} w \tag{22}$$

More over there exist $a, b > 0$ *such that*

$$u = aw_+ + bw_-$$

With $w_+ = sup(w,0)$ and $w_- = sup(-w,0)$.

PROOF. *Step1.* Applying Lemma 5 to $w_+ = sup(w,0)$ and $w_- = sup(-w,0)$, we get the existence of $a, b > 0$ such that

$$u = aw_+ + bw_-.$$

Now by Lemma 1, $w_+, w_- \in L^\infty$ i.e. $u \in L^\infty$, $u^{N-2} \in L^\infty$, then

$$h = S_g - \mu_2 u^{N-2} \in L^p$$

and from Theorem 12, we obtain

$$w \in H_2^p \subset C^{1-[n/p],\beta}.$$

Step 2. If $\mu_2 = \mu_1$, we see that $|w|$ is a minimizer to the functional associated to μ_1, then $|w|$ satisfies the same equation as v and Theorem 12 shows that $|w| = w \in H_2^p \subset\subset C^{1-[n/p],\beta}$ that is $|w| > 0$ everywhere, which contradicts the condition (9) in Proposition 3 , then

$$\mu_2 > \mu_1.$$

Step3. The solution w of the equation (22) changes sign. Since if it does not, we may assume that $w \geq 0$, by step2 the inequality in (20) is strict and by Lemma (5) we have the equality: a contradiction. □

REMARK 3. *Step1 shows that* u *is not necessarily in* $H_2^p(M)$ *and by the way the minimizing metric is not in* $H_2^p(M, T^*M \otimes T^*M)$ *contrary to the Yamabe invariant with singularities.*

References

[1] B. Ammann, E.Hunmbert. The second Yamabe invariant, J. Funct. Anal., 235 (2006),2, 377-412.

[2] T. Aubin. Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl., 55 (1976), 269-296.

[3] M.Benalili, H.Boughazi. On the second Paneitz-Branson invariant, Houston, J. Math.,36, (2010), 2, 393-420.

[4] E.Hebey. Introduction à l'analyse non linéaire sur les variétés, Diderot Editeur Arts Sciences (1997).

[5] F. Madani. Le problème de Yamabe avec singularités. Bull. Sci. Math. 132 (2008), 7, 575-591.

[6] R. Shoen. Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differential Geom. 20 (1984), 2, 479-495.

[7] R. Shoen, S.T.Yau. Conformally flat manifolds, Kleinian groupes and scalar curvature. Invent. Math, 9247-71, 1988.

[8] N.S.Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Pisa 32,3, (1968)

[9] H.Yamabe. On a deformation of Riemannian structures on compact manifolds. Osaka Math. J. 12 (1960) 21-37.

E-mail address: m_benalili@mail.univ-tlemcen.dz

Current address: Université Aboubekr Belkaïd, Faculty of Sciences, Dept. of Math. B.P. 119, Tlemcen, Algeria.